FEKETE-SZEGÖ PROBLEM AND SECOND HANKEL DETERMINANT FOR A CLASS OF BI-UNIVALENT FUNCTIONS

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Abstract. In this sequel to the recent work (see Azizi et al., 2015), we investigate a subclass of analytic and bi-univalent functions in the open unit disk. We obtain bounds for initial coefficients, the Fekete-Szegö inequality and the second Hankel determinant inequality for functions belonging to this subclass. We also discuss some new and known special cases, which can be deduced from our results.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disc $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and let $\mathcal{S}$ denote the class of functions in $\mathcal{A}$ that are univalent in $\mathbb{U}$.

For two functions f and g, analytic in $\mathbb{U}$, we say that the function f is subordinate to g in $\mathbb{U}$, and write $f \prec g$, if there exists a Schwarz function w, analytic in $\mathbb{U}$, with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$; $z, w \in \mathbb{U}$. In particular, if the function g is univalent in $\mathbb{U}$, the above subordination is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

Let φ be an analytic and univalent function with positive real part in $\mathbb{U}$, $\varphi(0) = 1$, $\varphi'(0) > 0$ and φ maps the unit disk $\mathbb{U}$ onto a region starlike with respect to 1 and symmetric with respect to the real axis. The Taylor's series expansion of such function is

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \ldots, \tag{1.2}$$

where all coefficients are real and $B_1 > 0$. Throughout this paper we assume that the function φ satisfies the above conditions unless otherwise stated.

By $\mathcal{S}^*(\varphi)$ and $\mathcal{K}(\varphi)$ we denote the following classes:

$$\mathcal{S}^*(\varphi) := \left\{ f \in \mathcal{S} : \frac{zf'(z)}{f(z)} \prec \varphi(z);\ z \in \mathbb{U} \right\}$$

and

$$\mathcal{K}(\varphi) := \left\{ f \in \mathcal{S} : 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z);\ z \in \mathbb{U} \right\}.$$

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The classes $\mathcal{S}^*(\varphi)$ and $\mathcal{K}(\varphi)$ are the extensions of a classical set of starlike and convex functions (e.g. see Ma and Minda [20]). For $0 \le \beta < 1$, the classes $\mathcal{S}^*(\beta) := \mathcal{S}^*\left(\frac{1+(1-2\beta)z}{1+z}\right)$ and $\mathcal{K}(\beta) := \mathcal{K}\left(\frac{1+(1-2\beta)z}{1+z}\right)$ are starlike and convex functions of order β.

It is well known (e.g. see Duren [12]) that every function $f \in \mathcal{S}$ has an inverse map f^{-1}, defined by $f^{-1}(f(z)) = z$, $z \in \mathbb{U}$ and $f(f^{-1}(w)) = w$, $(|w| < r_0(f);\ r_0(f) \geqq \frac{1}{4})$, where

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \dots . \tag{1.3}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. We let σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1.1). A function f is said to be bi-starlike of Ma-Minda type or bi-convex of Ma-Minda type if both f and f^{-1} are, respectively, of Ma-Minda starlike or convex type. These classes are denoted, respectively, by $\mathcal{S}^*_\sigma(\varphi)$ and $\mathcal{K}_\sigma(\varphi)$ (see [3]). For $0 \leqq \beta < 1$, a function $f \in \sigma$ is in the class $S^*_\sigma(\beta)$ of bi-starlike functions of order β, or $\mathcal{K}_\sigma(\beta)$ of bi-convex functions of order β if both f and its inverse map f^{-1} are, respectively, starlike or convex of order β. For a history and examples of functions which are (or which are not) in the class σ, together with various other properties of subclasses of bi-univalent functions one can refer [3,6,7,14,22,24,28,29].

For integers $n \ge 1$ and $q \ge 1$, the $q-$th Hankel determinant, defined as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q-2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q-2} & \cdots & a_{n+2q-2} \end{vmatrix} \qquad (a_1 = 1).$$

The Hankel determinant plays an important role in the study of singularities (see [11]). This is also an important in the study of power series with integral coefficients [8, 11]. The Hankel determinants $H_2(1) = a_3 - a_2^2$ and $H_2(2) = a_2a_4 - a_3^2$ are well-known as Fekete-Szegö and second Hankel determinant functionals respectively. Further Fekete and Szegö [13] introduced the generalized functional $a_3 - \delta a_2^2$, where δ is some real number. In 1969, Keogh and Merkes [18] discussed the Fekete-Szegö problem for the classes starlike and convex functions. Recently, several authors have investigated upper bounds for the Hankel determinant of functions belonging to various subclasses of univalent functions [2, 9, 19, 21] and the references therein. On the other hand, Zaprawa [29, 30] extended the study of Fekete-Szegö problem to certain subclasses of bi-univalent function class σ. Following Zaprawa [29, 30], the Fekete-Szegö problem for functions belonging to various other subclasses of bi-univalent functions were considered in [17, 23]. Very recently, the upper bounds of $H_2(2)$ for the classes $S^*_\sigma(\beta)$ and $K_\sigma(\beta)$ were discussed by Deniz et al. [10]. Recently, Lee et al. [19] introduced the following class:

$$\mathcal{G}^\lambda(\varphi) := \left\{ f \in \mathcal{S} : (1-\lambda)f'(z) + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \varphi(z);\ z \in \mathbb{U} \right\}$$

and obtained the bound for the second Hankel determinant of functions in $\mathcal{G}^\lambda(\varphi)$. It is interesting to note that

$$\mathcal{G}^\lambda := \mathcal{G}^\lambda\left(\frac{1+z}{1-z}\right) = \left\{ f : f \in \mathcal{S} \text{ and } \Re\left((1-\lambda)f'(z) + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right)\right) > 0;\ z \in \mathbb{U} \right\}.$$

The class $\mathcal{G}^\lambda$ introduced by Al-Amiri and Reade [1]. The univalence of the functions in the class $\mathcal{G}^\lambda$ was investigated by Singh et al. [26, 27].

Motivated by the recent publications (especially [4, 10, 17, 24, 29, 30]), we define the following subclass of σ.

Definition 1.1. For $0 \leq \lambda \leq 1$ and $0 \leq \beta < 1$, a function $f \in \sigma$ given by (1.1) is said to be in the class $\mathcal{G}_\sigma^\lambda(\varphi)$ if the following conditions are satisfied:

$$(1-\lambda)f'(z) + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \varphi(z), \qquad 0 \leqq \lambda \leqq 1,\, z \in \mathbb{U}$$

and for $g = f^{-1}$ given by (1.3)

$$(1-\lambda)g'(w) + \lambda\left(1 + \frac{wg''(w)}{g'(w)}\right) \prec \varphi(w), \qquad 0 \leqq \lambda \leqq 1,\, w \in \mathbb{U}.$$

From among the many choices of φ and λ which would provide the following known subclasses:

(1) $\mathcal{G}_\sigma^0(\varphi) := \mathcal{H}_\sigma(\varphi)$ [3],
(2) $\mathcal{G}_\sigma^1(\varphi) := \mathcal{K}_\sigma(\varphi)$ [3],
(3) $\mathcal{G}_\sigma^\lambda(\frac{1+(1-2\beta)z}{1-z}) := \mathcal{G}_\sigma^\lambda(\beta) \quad (0 \leq \beta < 1)$ [4].
(4) $\mathcal{G}_\sigma^0(\frac{1+(1-2\beta)z}{1-z}) := \mathcal{H}_\sigma^\beta \quad (0 \leq \beta < 1)$ [28]
(5) $\mathcal{G}_\sigma^1(\frac{1+(1-2\beta)z}{1-z}) := \mathcal{K}_\sigma(\beta) \quad (0 \leq \beta < 1)$ [5].

In this paper we shall obtain the Fekete-Szegö inequalities for $\mathcal{G}_\sigma^\lambda(\varphi)$ as well as its special classes. Further, the second Hankel determinant obtained for the class $\mathcal{G}_\sigma^\lambda(\beta)$.

2. Initial Coefficient Bounds

Theorem 2.1. *If f given by (1.1) is in the class $\mathcal{G}_\sigma^\lambda(\varphi)$, then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{4B_1 + |(3-\lambda)B_1^2 - 4B_2|}} \tag{2.1}$$

and

$$|a_3| \leq \begin{cases} \left(1 - \frac{4}{3(1+\lambda)B_1}\right)\frac{B_1^3}{4B_1+|(3-\lambda)B_1^2-4B_2|} + \frac{B_1}{3(1+\lambda)}, & \text{if} \quad B_1 \geq \frac{4}{3(1+\lambda)}; \\ \frac{B_1}{3(1+\lambda)}, & \text{if} \quad B_1 < \frac{4}{3(1+\lambda)}. \end{cases} \tag{2.2}$$

Proof. Suppose that $u(z)$ and $v(z)$ are analytic in the unit disk $\mathbb{U}$ with $u(0) = v(0) = 0$, $|u(z)| < 1$, $|v(z)| < 1$ and

$$u(z) = b_1 z + \sum_{n=2}^{\infty} b_n z^n,\ v(z) = c_1 z + \sum_{n=2}^{\infty} c_n z^n, \qquad |z| < 1. \tag{2.3}$$

It is well known that

$$|b_1| \leq 1,\ |b_2| \leq 1 - |b_1|^2,\ |c_1| \leq 1,\ |c_2| \leq 1 - |c_1|^2. \tag{2.4}$$

By a simple calculation, we have

$$\varphi(u(z)) = 1 + B_1 b_1 z + (B_1 b_2 + B_2 b_1^2)z^2 + \ldots,\quad |z| < 1 \tag{2.5}$$

and

$$\varphi(v(w)) = 1 + B_1 c_1 w + (B_1 c_2 + B_2 c_1^2)w^2 + \ldots,\quad |w| < 1. \tag{2.6}$$

Let $f \in \mathcal{G}_\sigma^\lambda(\varphi)$. Then there are analytic functions $u, v : \mathbb{U} \to \mathbb{U}$ given by (2.3) such that

$$(1-\lambda)f'(z) + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right) = \varphi(u(z)) \tag{2.7}$$

and

$$(1-\lambda)g'(w) + \lambda\left(1 + \frac{wg''(w)}{g'(w)}\right) = \varphi(v(w)). \tag{2.8}$$

It follows from (2.5), (2.6), (2.7) and (2.8) that

$$2a_2 = B_1 b_1 \tag{2.9}$$

$$3(1+\lambda)a_3 - 4\lambda a_2^2 = B_1 b_2 + B_2 b_1^2 \tag{2.10}$$

$$-2a_2 = B_1 c_1 \tag{2.11}$$

$$2(\lambda+3)a_2^2 - 3(1+\lambda)a_3 = B_1 c_2 + B_2 c_1^2. \tag{2.12}$$

From (2.9) and (2.11), we get

$$b_1 = -c_1. \tag{2.13}$$

By adding (2.10) to (2.12), further, using (2.9) and (2.13), we have

$$(2(3-\lambda)B_1^2 - 8B_2)a_2^2 = B_1^3(b_2 + c_2). \tag{2.14}$$

In view of (2.13) and (2.14), together with (2.4), we get

$$|(2(3-\lambda)B_1^2 - 8B_2)a_2^2| \le 2B_1^3(1 - |b_1|^2). \tag{2.15}$$

Substituting (2.9) in (2.15) we obtain

$$|a_2| \le \frac{B_1\sqrt{B_1}}{\sqrt{4B_1 + |(3-\lambda)B_1^2 - 4B_2|}}. \tag{2.16}$$

By subtracting (2.12) from (2.10) and in view of (2.13), we get

$$6(1+\lambda)a_3 = 6(1+\lambda)a_2^2 + B_1(b_2 - c_2). \tag{2.17}$$

From (2.4), (2.9), (2.13) and (2.17), it follows that

$$\begin{aligned} |a_3| &\le |a_2|^2 + \frac{B_1}{6(1+\lambda)}(|b_2| + |c_2|) \\ &\le |a_2|^2 + \frac{B_1}{3(1+\lambda)}(1 - |b_1|^2) \\ &= \left(1 - \frac{4}{3(1+\lambda)B_1}\right)|a_2|^2 + \frac{B_1}{3(1+\lambda)}. \end{aligned} \tag{2.18}$$

Substituting (2.16) in (2.18) we obtain the desired inequality (2.2). □

Remark 2.1. For $\lambda = 0$, the results obtained in the Theorem 2.1 are coincide with results in [24, Theorem 2.1, p.230].

Corollary 2.1. *Let $f \in \mathcal{K}_\sigma(\varphi)$. Then*

$$|a_2| \le \frac{B_1\sqrt{B_1}}{\sqrt{4B_1 + |2B_1^2 - 4B_2|}} \tag{2.19}$$

and

$$|a_3| \leq \begin{cases} \left(1-\frac{2}{3B_1}\right)\frac{B_1^3}{4B_1+|2B_1^2-4B_2|}+\frac{B_1}{6} & ;B_1 \geq \frac{2}{3}; \\ \frac{B_1}{3(1+\lambda)} & ;B_1 < \frac{2}{3}. \end{cases} \tag{2.20}$$

3. Fekete-Szegö inequalities

In order to derive our result, we shall need the following lemma.

Lemma 3.1. *(see [12] or [16]) Let* $p(z) = 1 + p_1 z + p_2 z^2 + \cdots \in \mathcal{P}$, *where* $\mathcal{P}$ *is the family of all functions* p, *analytic in* $\mathbb{U}$, *for which* $\Re\{p(z)\} > 0$, $z \in \mathbb{U}$. *Then*

$$|p_n| \leqq 2; \qquad n = 1, 2, 3, ...,$$

and

$$\left|p_2 - \frac{1}{2}p_1^2\right| \leq 2 - \frac{1}{2}|p_1|^2.$$

Theorem 3.1. *Let* f *of the form (1.1) be in* $\mathcal{G}_\sigma^\lambda(\varphi)$. *Then*

$$|a_2| \leq \begin{cases} \sqrt{\frac{B_1}{3-\lambda}}, & \text{if} \quad |B_2| \leq B_1; \\ \sqrt{\frac{|B_2|}{3-\lambda}}, & \text{if} \quad |B_2| \geq B_1 \end{cases} \tag{3.1}$$

and

$$\left|a_3 - \frac{4\lambda}{3+3\lambda}a_2^2\right| \leq \begin{cases} \frac{B_1}{3+3\lambda}, & \text{if} \quad |B_2| \leq B_1; \\ \frac{|B_2|}{3+3\lambda}, & \text{if} \quad |B_2| \geq B_1. \end{cases} \tag{3.2}$$

Proof. Since $f \in \mathcal{G}_\sigma^\lambda(\varphi)$, there exist two analytic functions $r, s : \mathbb{U} \to \mathbb{U}$, with $r(0) = 0 = s(0)$, such that

$$(1-\lambda)f'(z) + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right) = \varphi(r(z)) \tag{3.3}$$

and

$$(1-\lambda)g'(w) + \lambda\left(1 + \frac{wg''(w)}{g'(w)}\right) = \varphi(s(w)). \tag{3.4}$$

Define the functions p and q by

$$p(z) = \frac{1+r(z)}{1-r(z)} = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \ldots$$

and

$$q(w) = \frac{1+s(w)}{1-s(w)} = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \ldots$$

or equivalently,

$$r(z) = \frac{p(z)-1}{p(z)+1} = \frac{1}{2}\left(p_1 z + \left(p_2 - \frac{p_1^2}{2}\right)z^2 + \left(p_3 + \frac{p_1}{2}\left(\frac{p_1^2}{2} - p_2\right) - \frac{p_1 p_2}{2}\right)z^3 + \ldots\right) \tag{3.5}$$

and

$$s(w)=\frac{q(w)-1}{q(w)+1}=\frac{1}{2}\left(q_1w+\left(q_2-\frac{q_1^2}{2}\right)w^2+\left(q_3+\frac{q_1}{2}\left(\frac{q_1^2}{2}-q_2\right)-\frac{q_1q_2}{2}\right)w^3+\dots\right). \tag{3.6}$$

Using (3.5) and (3.6) in (3.3) and (3.4), we have

$$(1-\lambda)f'(z)+\lambda\left(1+\frac{zf''(z)}{f'(z)}\right)=\varphi\left(\frac{p(z)-1}{p(z)+1}\right) \tag{3.7}$$

and

$$(1-\lambda)g'(w)+\lambda\left(1+\frac{wg''(w)}{g'(w)}\right)=\varphi\left(\frac{q(w)-1}{q(w)+1}\right). \tag{3.8}$$

Again using (3.5) and (3.6) along with (1.2), it is evident that

$$\varphi\left(\frac{p(z)-1}{p(z)+1}\right)=1+\frac{1}{2}B_1p_1z+\left(\frac{1}{2}B_1\left(p_2-\frac{1}{2}p_1^2\right)+\frac{1}{4}B_2p_1^2\right)z^2+\dots \tag{3.9}$$

and

$$\varphi\left(\frac{q(w)-1}{q(w)+1}\right)=1+\frac{1}{2}B_1q_1w+\left(\frac{1}{2}B_1\left(q_2-\frac{1}{2}q_1^2\right)+\frac{1}{4}B_2q_1^2\right)w^2+\dots. \tag{3.10}$$

It follows from (3.7), (3.8), (3.9) and (3.10) that

$$2a_2 = \frac{1}{2}B_1p_1$$

$$3(1+\lambda)a_3-4\lambda a_2^2 = \frac{1}{2}B_1\left(p_2-\frac{1}{2}p_1^2\right)+\frac{1}{4}B_2p_1^2 \tag{3.11}$$

$$-2a_2 = \frac{1}{2}B_1q_1$$

$$2(\lambda+3)a_2^2-3(1+\lambda)a_3 = \frac{1}{2}B_1\left(q_2-\frac{1}{2}q_1^2\right)+\frac{1}{4}B_2q_1^2. \tag{3.12}$$

Dividing (3.11) by $3+3\lambda$ and taking the absolute values we obtain

$$\left|a_3-\frac{4\lambda}{3+3\lambda}a_2^2\right|\le\frac{B_1}{6+6\lambda}\left|p_2-\frac{1}{2}p_1^2\right|+\frac{|B_2|}{12+12\lambda}|p_1|^2.$$

Now applying Lemma 3.1, we have

$$\left|a_3-\frac{4\lambda}{3+3\lambda}a_2^2\right|\le\frac{B_1}{3+3\lambda}+\frac{|B_2|-B_1}{12+12\lambda}|p_1|^2.$$

Therefore

$$\left|a_3-\frac{4\lambda}{3+3\lambda}a_2^2\right|\le\begin{cases}\frac{B_1}{3+3\lambda}, & \text{if} \quad |B_2|\le B_1;\\[2ex] \frac{|B_2|}{3+3\lambda}, & \text{if} \quad |B_2|\ge B_1.\end{cases}$$

Adding (3.11) and (3.12), we have

$$(6-2\lambda)a_2^2=\frac{B_1}{2}(p_2+q_2)-\frac{(B_1-B_2)}{4}(p_1^2+q_1^2). \tag{3.13}$$

Dividing (3.13) by $6-2\lambda$ and taking the absolute values we obtain

$$|a_2|^2 \leq \frac{1}{6-2\lambda}\left[\frac{B_1}{2}\left|p_2-\frac{1}{2}p_1^2\right| + \frac{|B_2|}{4}|p_1|^2 + \frac{B_1}{2}\left|q_2-\frac{1}{2}q_1^2\right| + \frac{|B_2|}{4}|q_1|^2\right].$$

Once again, apply Lemma 3.1 to obtain

$$|a_2|^2 \leq \frac{1}{6-2\lambda}\left[\frac{B_1}{2}\left(2-\frac{1}{2}|p_1|^2\right) + \frac{|B_2|}{4}|p_1|^2 + \frac{B_1}{2}\left(2-\frac{1}{2}|q_1|^2\right) + \frac{|B_2|}{4}|q_1|^2\right].$$

Upon simplification we obtain

$$|a_2|^2 \leq \frac{1}{6-2\lambda}\left[2B_1 + \frac{|B_2|-B_1}{2}\left(|p_1|^2+|q_1|^2\right)\right].$$

Therefore

$$|a_2| \leq \begin{cases} \sqrt{\frac{B_1}{3-\lambda}}, & \text{if } \quad |B_2| \leq B_1; \\ \sqrt{\frac{|B_2|}{3-\lambda}}, & \text{if } \quad |B_2| \geq B_1 \end{cases}$$

which completes the proof. $\square$

Remark 3.1. Taking

$$\varphi(z) = \left(\frac{1+z}{1-z}\right)^{\beta} = 1 + 2\beta z + 2\beta^2 z^2 + \ldots, \qquad 0 < \beta \leq 1 \tag{3.14}$$

the inequalities (3.1) and (3.2) become

$$|a_2| \leq \sqrt{\frac{2\beta}{3-\lambda}} \qquad \text{and} \qquad \left|a_3 - \frac{4\lambda}{3+3\lambda}a_2^2\right| \leq \frac{2\beta}{3+3\lambda}. \tag{3.15}$$

For

$$\varphi(z) = \frac{1+(1-2\beta)z}{1-z} = 1 + 2(1-\beta)z + 2(1-\beta)z^2 + \ldots, \qquad 0 \leq \beta < 1 \tag{3.16}$$

the inequalities (3.1) and (3.2) become

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{3-\lambda}} \qquad \text{and} \qquad \left|a_3 - \frac{4\lambda}{3-\lambda}a_2^2\right| \leq \frac{2(1-\beta)}{3+3\lambda}. \tag{3.17}$$

4. Bounds for the second Hankel determinant of $\mathcal{G}_{\sigma}^{\lambda}(\beta)$

Next we state the following lemmas to establish the desired bounds in our study.

Lemma 4.1. *[25] If the function $p \in \mathcal{P}$ is given by the series*

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots, \tag{4.1}$$

then the following sharp estimate holds:

$$|p_n| \leq 2, \qquad n = 1, 2, \cdots. \tag{4.2}$$

Lemma 4.2. *[15] If the function $p \in \mathcal{P}$ is given by the series (4.1), then*

$$\begin{aligned} 2c_2 &= c_1^2 + x(4-c_1^2) \\ 4c_3 &= c_1^3 + 2c_1(4-c_1^2)x - c_1(4-c_1^2)x^2 + 2(4-c_1^2)(1-|x|^2)z \end{aligned}$$

for some x, z with $|x| \le 1$ and $|z| \le 1$.

The following theorem provides a bound for the second Hankel determinant of the functions in the class $\mathcal{G}_\sigma^\lambda(\beta)$.

Theorem 4.1. *Let f of the form (1.1) be in $\mathcal{G}_\sigma^\lambda(\beta)$. Then*

$$|a_2a_4 - a_3^2| \le \begin{cases} \frac{(1-\beta)^2}{2(1+2\lambda)}\left[(2-\lambda)(1-\beta)^2+1\right]\ ; \\ \qquad\qquad \beta \in \left[0, 1 - \frac{(1+2\lambda)+\sqrt{(1+2\lambda)^2+18(1+\lambda)^2(2-\lambda)}}{6(1+\lambda)(2-\lambda)}\right] \\ \frac{(1-\beta)^2}{72(1+2\lambda)}\left(\begin{array}{c} 36[8(1+2\lambda)(2-\lambda)-(1+2\lambda)^2](1-\beta)^2 \\ \dfrac{-324(1+\lambda)(1+2\lambda)(1-\beta)+288(1+2\lambda)-729(1+\lambda)^2}{9(1+\lambda)^2(2-\lambda)(1-\beta)^2-6(1+\lambda)(1+2\lambda)(1-\beta)} \\ +8(1+2\lambda)-18(1+\lambda)^2 \end{array}\right) ; \\ \qquad\qquad \beta \in \left(1 - \frac{(1+2\lambda)+\sqrt{(1+2\lambda)^2+18(1+\lambda)^2(2-\lambda)}}{6(1+\lambda)(2-\lambda)}, 1\right). \end{cases}$$

Proof. Let $f \in \mathcal{G}_\sigma^\lambda(\beta)$. Then

$$(1-\lambda)f'(z) + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right) = \beta + (1-\beta)p(z) \tag{4.3}$$

and

$$(1-\lambda)g'(w) + \lambda\left(1 + \frac{wg''(w)}{g'(w)}\right) = \beta + (1-\beta)q(w), \tag{4.4}$$

where $p, q \in \mathcal{P}$ and defined by

$$p(z) = 1 + c_1z + c_2z^2 + c_3z^3 + \dots \tag{4.5}$$

and

$$q(z) = 1 + d_1w + d_2w^2 + d_3w^3 + \dots. \tag{4.6}$$

It follows from (4.3), (4.4), (4.5) and (4.6) that

$$2a_2 = (1-\beta)c_1 \tag{4.7}$$

$$3(1+\lambda)a_3 - 4\lambda a_2^2 = (1-\beta)c_2 \tag{4.8}$$

$$4(1+2\lambda)a_4 - 18\lambda a_2a_3 + 8\lambda a_2^3 = (1-\beta)c_3 \tag{4.9}$$

and

$$-2a_2 = (1-\beta)d_1 \tag{4.10}$$

$$2(3+\lambda)a_2^2 - 3(1+\lambda)a_3 = (1-\beta)d_2 \tag{4.11}$$

$$2(10+11\lambda)a_2a_3 - 4(5+3\lambda)a_2^3 - 4(1+2\lambda)a_4 = (1-\beta)d_3. \tag{4.12}$$

From (4.7) and (4.10), we find that

$$c_1 = -d_1 \tag{4.13}$$

and

$$a_2 = \frac{1-\beta}{2} c_1. \tag{4.14}$$

Now, from (4.8), (4.11) and (4.14), we have

$$a_3 = \frac{(1-\beta)^2}{4} c_1^2 + \frac{1-\beta}{6(1+\lambda)}(c_2 - d_2). \tag{4.15}$$

Also, from (4.9) and (4.12), we find that

$$a_4 = \frac{5\lambda(1-\beta)^3}{16(1+2\lambda)} c_1^3 + \frac{5(1-\beta)^2}{24(1+\lambda)} c_1(c_2 - d_2) + \frac{1-\beta}{8(1+2\lambda)}(c_3 - d_3). \tag{4.16}$$

Then, we can establish that

$$\begin{aligned} |a_2 a_4 - a_3^2| &= \left| \frac{(\lambda-2)(1-\beta)^4}{32(1+2\lambda)} c_1^4 + \frac{(1-\beta)^3}{48(1+\lambda)} c_1^2 (c_2 - d_2) \right. \\ &\quad \left. + \frac{(1-\beta)^2}{16(1+2\lambda)} c_1(c_3 - d_3) - \frac{(1-\beta)^2}{36(1+\lambda)^2}(c_2 - d_2)^2 \right|. \end{aligned} \tag{4.17}$$

According to Lemma 4.2 and (4.13), we write

$$c_2 - d_2 = \frac{(4-c_1^2)}{2}(x-y) \tag{4.18}$$

$$\begin{aligned} c_3 - d_3 &= \frac{c_1^3}{2} + \frac{c_1(4-c_1^2)(x+y)}{2} - \frac{c_1(4-c_1^2)(x^2+y^2)}{4} \\ &\quad + \frac{(4-c_1^2)[(1-|x|^2)z - (1-|y|^2)w]}{2} \end{aligned} \tag{4.19}$$

for some x, y, z and w with $|x| \le 1$, $|y| \le 1$, $|z| \le 1$ and $|w| \le 1$. Using (4.18) and (4.19) in (4.17), we have

$$\begin{aligned} |a_2 a_4 - a_3^2| &= \left| \frac{(\lambda-2)(1-\beta)^4 c_1^4}{32(1+2\lambda)} + \frac{(1-\beta)^3 c_1^2 (4-c_1^2)(x-y)}{96(1+\lambda)} + \frac{(1-\beta)^2 c_1}{16(1+2\lambda)} \right. \\ &\quad \times \left[\frac{c_1^3}{2} + \frac{c_1(4-c_1^2)(x+y)}{2} - \frac{c_1(4-c_1^2)(x^2+y^2)}{4} \right. \\ &\quad \left. \left. + \frac{(4-c_1^2)[(1-|x|^2)z - (1-|y|^2)w]}{2} \right] - \frac{(1-\beta)^2(4-c_1^2)^2}{144(1+\lambda)^2}(x-y)^2 \right| \\ &\le \frac{(2-\lambda)(1-\beta)^4}{32(1+2\lambda)} c_1^4 + \frac{(1-\beta)^2 c_1^4}{32(1+2\lambda)} + \frac{(1-\beta)^2 c_1 (4-c_1^2)}{16(1+2\lambda)} \\ &\quad + \left[\frac{(1-\beta)^3 c_1^2 (4-c_1^2)}{96(1+\lambda)} + \frac{(1-\beta)^2 c_1^2 (4-c_1^2)}{32(1+2\lambda)} \right] (|x| + |y|) \\ &\quad + \left[\frac{(1-\beta)^2 c_1^2 (4-c_1^2)}{64(1+2\lambda)} - \frac{(1-\beta)^2 c_1 (4-c_1^2)}{32(1+2\lambda)} \right] (|x|^2 + |y|^2) \\ &\quad + \frac{(1-\beta)^2 (4-c_1^2)^2}{144(1+\lambda)^2} (|x| + |y|)^2. \end{aligned}$$

Since $p \in \mathcal{P}$, so $|c_1| \leq 2$. Letting $c_1 = c$, we may assume without restriction that $c \in [0, 2]$. Thus, for $\gamma_1 = |x| \leq 1$ and $\gamma_2 = |y| \leq 1$, we obtain

$$|a_2a_4 - a_3^2| \leq T_1 + T_2(\gamma_1 + \gamma_2) + T_3(\gamma_1^2 + \gamma_2^2) + T_4(\gamma_1 + \gamma_2)^2 = F(\gamma_1, \gamma_2),$$

$$\begin{aligned} T_1 &= T_1(c) = \frac{(2-\lambda)(1-\beta)^4}{32(1+2\lambda)}c^4 + \frac{(1-\beta)^2c^4}{32(1+2\lambda)} + \frac{(1-\beta)^2c(4-c^2)}{16(1+2\lambda)} \geq 0 \\ T_2 &= T_2(c) = \frac{(1-\beta)^3c^2(4-c^2)}{96(1+\lambda)} + \frac{(1-\beta)^2c^2(4-c^2)}{32(1+2\lambda)} \geq 0 \\ T_3 &= T_3(c) = \frac{(1-\beta)^2c^2(4-c^2)}{64(1+2\lambda)} - \frac{(1-\beta)^2c(4-c^2)}{32(1+2\lambda)} \leq 0 \\ T_4 &= T_4(c) = \frac{(1-\beta)^2(4-c^2)^2}{144(1+\lambda)^2} \geq 0. \end{aligned}$$

Now we need to maximize $F(\gamma_1, \gamma_2)$ in the closed square $\mathbb{S} := \{(\gamma_1, \gamma_2) : 0 \leq \gamma_1 \leq 1, 0 \leq \gamma_2 \leq 1\}$ for $c \in [0, 2]$. We must investigate the maximum of $F(\gamma_1, \gamma_2)$ according to $c \in (0, 2)$, $c = 0$ and $c = 2$ taking into account the sign of $F_{\gamma_1\gamma_1}F_{\gamma_2\gamma_2} - (F_{\gamma_1\gamma_2})^2$.

Firstly, let $c \in (0, 2)$. Since $T_3 < 0$ and $T_3 + 2T_4 > 0$ for $c \in (0, 2)$, we conclude that

$$F_{\gamma_1\gamma_1}F_{\gamma_2\gamma_2} - (F_{\gamma_1\gamma_2})^2 < 0.$$

Thus, the function F cannot have a local maximum in the interior of the square $\mathbb{S}$. Now, we investigate the maximum of F on the boundary of the square $\mathbb{S}$.

For $\gamma_1 = 0$ and $0 \leq \gamma_2 \leq 1$ (similarly $\gamma_2 = 0$ and $0 \leq \gamma_1 \leq 1$) we obtain

$$F(0, \gamma_2) = G(\gamma_2) = T_1 + T_2\gamma_2 + (T_3 + T_4)\gamma_2^2.$$

(i) The case $T_3 + T_4 \geq 0$: In this case for $0 < \gamma_2 < 1$ and any fixed c with $0 < c < 2$, it is clear that $G'(\gamma_2) = 2(T_3 + T_4)\gamma_2 + T_2 > 0$, that is, $G(\gamma_2)$ is an increasing function. Hence, for fixed $c \in (0, 2)$, the maximum of $G(\gamma_2)$ occurs at $\gamma_2 = 1$ and

$$\max G(\gamma_2) = G(1) = T_1 + T_2 + T_3 + T_4.$$

(ii) The case $T_3 + T_4 < 0$: Since $T_2 + 2(T_3 + T_4) \geq 0$ for $0 < \gamma_2 < 1$ and any fixed c with $0 < c < 2$, it is clear that $T_2 + 2(T_3 + T_4) < 2(T_3 + T_4)\gamma_2 + T_2 < T_2$ and so $G'(\gamma_2) > 0$. Hence for fixed $c \in (0, 2)$, the maximum of $G(\gamma_2)$ occurs at $\gamma_2 = 1$ and also for $c = 2$ we obtain

$$F(\gamma_1, \gamma_2) = \frac{(1-\beta)^2}{2(1+2\lambda)}\left[(2-\lambda)(1-\beta)^2 + 1\right]. \tag{4.20}$$

Taking into account the value (4.20) and the cases i and ii, for $0 \leq \gamma_2 < 1$ and any fixed c with $0 \leq c \leq 2$ we have

$$\max G(\gamma_2) = G(1) = T_1 + T_2 + T_3 + T_4.$$

For $\gamma_1 = 1$ and $0 \leq \gamma_2 \leq 1$ (similarly $\gamma_2 = 1$ and $0 \leq \gamma_1 \leq 1$), we obtain

$$F(1, \gamma_2) = H(\gamma_2) = (T_3 + T_4)\gamma_2^2 + (T_2 + 2T_4)\gamma_2 + T_1 + T_2 + T_3 + T_4.$$

Similarly, to the above cases of $T_3 + T_4$, we get that

$$\max H(\gamma_2) = H(1) = T_1 + 2T_2 + 2T_3 + 4T_4.$$

Since $G(1) \leq H(1)$ for $c \in (0, 2)$, $\max F(\gamma_1, \gamma_2) = F(1, 1)$ on the boundary of the square $\mathbb{S}$. Thus the maximum of F occurs at $\gamma_1 = 1$ and $\gamma_2 = 1$ in the closed square $\mathbb{S}$.

Let $K : (0, 2) \to \mathbb{R}$

$$K(c) = \max F(\gamma_1, \gamma_2) = F(1, 1) = T_1 + 2T_2 + 2T_3 + 4T_4. \tag{4.21}$$

Substituting the values of T_1, T_2, T_3 and T_4 in the function K defined by (4.21), yields

$$\begin{aligned} K(c) = \frac{(1-\beta)^2}{288(1+\lambda)^2(1+2\lambda)} & \left\{ \left[9(1-\beta)^2(1+\lambda)^2(2-\lambda) \right.\right. \\ & \left. -6(1-\beta)(1+\lambda)(1+2\lambda) - 18(1+\lambda)^2 + 8(1+2\lambda) \right] c^4 \\ & + \left[24(1-\beta)(1+\lambda)(1+2\lambda) + 108(1+\lambda)^2 - 64(1+2\lambda) \right] c^2 \\ & \left. +128(1+2\lambda) \right\}. \end{aligned}$$

Assume that $K(c)$ has a maximum value in an interior of $c \in (0, 2)$, by elementary calculation, we find

$$\begin{aligned} K'(c) = \frac{(1-\beta)^2}{72(1+\lambda)^2(1+2\lambda)} & \left\{ \left[9(1-\beta)^2(1+\lambda)^2(2-\lambda) \right.\right. \\ & \left. -6(1-\beta)(1+\lambda)(1+2\lambda) - 18(1+\lambda)^2 + 8(1+2\lambda) \right] c^3 \\ & \left. + \left[12(1-\beta)(1+\lambda)(1+2\lambda) + 54(1+\lambda)^2 - 32(1+2\lambda) \right] c \right\}. \end{aligned}$$

After some calculations we concluded the following cases:

Case 4.1. Let

$$[9(1-\beta)^2(1+\lambda)^2(2-\lambda) - 6(1-\beta)(1+\lambda)(1+2\lambda) - 18(1+\lambda)^2 + 8(1+2\lambda)] \geq 0,$$

that is,

$$\beta \in \left[0, 1 - \frac{(1+2\lambda) + \sqrt{(1+2\lambda)^2 + (2-\lambda)[18(1+\lambda)^2 - 8(1+2\lambda)]}}{3(1+\lambda)(2-\lambda)} \right].$$

Therefore $K'(c) > 0$ for $c \in (0, 2)$. Since K is an increasing function in the interval $(0, 2)$, maximum point of K must be on the boundary of $c \in [0, 2]$, that is, $c = 2$. Thus, we have

$$\max_{0<c<2} K(c) = K(2) = \frac{(1-\beta)^2}{2(1+2\lambda)} \left[(2-\lambda)(1-\beta)^2 + 1 \right].$$

Case 4.2. Let

$$[9(1-\beta)^2(1+\lambda)^2(2-\lambda) - 6(1-\beta)(1+\lambda)(1+2\lambda) - 18(1+\lambda)^2 + 8(1+2\lambda)] < 0,$$

that is,

$$\beta \in \left[1 - \frac{(1+2\lambda) + \sqrt{(1+2\lambda)^2 + (2-\lambda)[18(1+\lambda)^2 - 8(1+2\lambda)]}}{3(1+\lambda)(2-\lambda)}, 1 \right].$$

Then $K'(c) = 0$ implies the real critical point $c_{0_1} = 0$ or

$$c_{0_2} = \sqrt{\frac{-12(1+\lambda)(1+2\lambda)(1-\beta) - 54(1+\lambda)^2 + 32(1+2\lambda)}{9(1-\beta)^2(1+\lambda)^2(2-\lambda) - 6(1-\beta)(1+\lambda)(1+2\lambda) - 18(1+\lambda)^2 + 8(1+2\lambda)}}.$$

When

$$\beta \in \left(1 - \frac{(1+2\lambda)+\sqrt{(1+2\lambda)^2+(2-\lambda)[18(1+\lambda)^2-8(1+2\lambda)]}}{3(1+\lambda)(2-\lambda)}, 1 - \frac{(1+2\lambda)+\sqrt{(1+2\lambda)^2+18(1+\lambda)^2(2-\lambda)]}}{6(1+\lambda)(2-\lambda)} \right].$$

We observe that $c_{0_2} \geq 2$, that is, c_{0_2} is out of the interval $(0, 2)$. Therefore, the maximum value of $K(c)$ occurs at $c_{0_1} = 0$ or $c = c_{0_2}$ which contradicts our assumption of having the maximum value at the interior point of $c \in [0, 2]$. Since K is an increasing function in the interval $(0, 2)$, maximum point of K must be on the boundary of $c \in [0, 2]$ that is $c = 2$. Thus, we have

$$\max_{0\leq c\leq 2} K(c) = K(2) = \frac{(1-\beta)^2}{2(1+2\lambda)}[1 + (2-\lambda)(1-\beta)^2].$$

When $\beta \in \left(1 - \frac{(1+2\lambda)+\sqrt{(1+2\lambda)^2+18(1+\lambda)^2(2-\lambda)]}}{6(1+\lambda)(2-\lambda)}, 1\right)$, we observe that $c_{0_2} < 2$, that is, c_{0_2} is an interior of the interval $[0, 2]$. Since $K''(c_{0_2}) < 0$, the maximum value of $K(c)$ occurs at $c = c_{0_2}$. Thus, we have

$$\begin{aligned}\max_{0\leq c\leq 2} K(c) &= K(c_{0_2}) \\ &= \frac{(1-\beta)^2}{72(1+2\lambda)}\left(\frac{\begin{array}{c}36[8(1+2\lambda)(2-\lambda)-(1+2\lambda)^2](1-\beta)^2 \\ -324(1+\lambda)(1+2\lambda)(1-\beta)+288(1+2\lambda)-729(1+\lambda)^2\end{array}}{\begin{array}{c}9(1+\lambda)^2(2-\lambda)(1-\beta)^2 \\ -6(1+\lambda)(1+2\lambda)(1-\beta)+8(1+2\lambda)-18(1+\lambda)^2\end{array}}\right).\end{aligned}$$

This completes the proof. □

Corollary 4.1. *Let f of the form (1.1) be in $\mathcal{H}_\sigma^\beta$. Then*

$$|a_2a_4 - a_3^2| \leq \begin{cases} \frac{(1-\beta)^2[1+2(1-\beta)^2]}{2}; & \beta \in \left[0, \frac{11-\sqrt{37}}{12}\right] \\ \frac{(1-\beta)^2[60\beta^2-84\beta-25]}{16(9\beta^2-15\beta+1)}; & \beta \in \left(\frac{11-\sqrt{37}}{12}, 1\right). \end{cases}$$

Corollary 4.2. *Let f of the form (1.1) be in $\mathcal{K}_\sigma(\beta)$. Then*

$$|a_2a_4 - a_3^2| \leq \frac{(1-\beta)^2}{24}\left[\frac{5\beta^2+8\beta-32}{3\beta^2-3\beta-4}\right].$$

Corollary 4.3. *Let f of the form (1.1) be in $\mathcal{H}_\sigma$. Then*

$$|a_2a_4 - a_3^2| \leq \frac{3}{2}.$$

Corollary 4.4. *Let f of the form (1.1) be in $\mathcal{K}_\sigma$. Then*

$$|a_2a_4 - a_3^2| \leq \frac{1}{3}.$$

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